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Magnetoconductance of small electrical networks

B R Bulka
Institute of Molecular Physics, Polish Academy of Sciences, Smoluchowskiego 17, 60–179 Poznań, Poland

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Abstract. An approach used to calculate the conductivity of small electrical networks in the presence of a magnetic field is presented. Boundary conditions are taken into account during the calculations of the Green function. The conductivity is determined from the Kubo formula exactly. The roles of inelastic scattering and topology of the system in a quantum interference process are considered.

1. Introduction

Recently, much interest has been devoted to the transport properties of small metallic systems (see [1] and references therein; see also [2]). The size of the samples is smaller than the inelastic scattering length of electrons. In such a case, quantum interference plays a crucial role in the electronic transport. There is a lack of self-averaging of physical quantities. Results differ from sample to sample; they depend on external conditions (the electrical and magnetic field). Theoretical approaches to the problem are usually in the framework of weak localisation [2], in which electron scattering due to impurities and quantum interference determine the transport properties.

In the present work, we focus our attention on a pure system (i.e. without impurities). Scattering occurs only on nodes of the network. We investigate the role of the geometry of the electrical network in quantum interference processes. In § 2, we present our approach which is based on the Kubo formula for the conductivity. Further calculations are analytical and exact. In § 3, this approach is used for electrical networks with various geometries: a single ring, a ring with attached leads, a single wire, and a wire with an attached bubble. Although our model is very simple, we think that it may be useful in understanding experimental data obtained on very small metallic systems [1].

2. Conductivity of a confined system in the Boltzmann approximation

We investigate the dependence of the conductivity for electrical networks with different geometries. It is assumed that the system consists of thin pure wires. Impurities and defects play no role, and the elastic scattering length l_e for electrons is much larger than the inelastic scattering length l_{in}. In such a case, electrons can be considered as quasi-particles with a lifetime $\tau = l_{in}/v_F$ (v_F is the Fermi velocity), and the conductivity σ may be determined in the Boltzmann approximation. From the linear response theory, one can obtain

$$\sigma(E_F) = \frac{e^2/h}{4m^2V}\int dE \left(-\frac{\partial f}{\partial E}\right)\int d\boldsymbol{r}\int d\boldsymbol{r}'\, J(\boldsymbol{r},\boldsymbol{r}';E) \tag{1}$$

where

$$\begin{aligned} J(\boldsymbol{r},\boldsymbol{r}';E) = & -2(\partial/\partial z)(\partial/\partial z)\{\mathrm{Im}[G(\boldsymbol{r},\boldsymbol{r}';E)]\}\,\mathrm{Im}\,G(\boldsymbol{r}',\boldsymbol{r};E) \\ & + (\partial/\partial z)\{\mathrm{Im}[G(\boldsymbol{r},\boldsymbol{r}';E)]\}\,(\partial/\partial z')\{\mathrm{Im}[G(\boldsymbol{r}',\boldsymbol{r};E)]\} \\ & + (\partial/\partial z')\{\mathrm{Im}[G(\boldsymbol{r},\boldsymbol{r}';E)]\}\,(\partial/\partial z)\{\mathrm{Im}[G(\boldsymbol{r}',\boldsymbol{r};E)]\}. \end{aligned}$$

Here m denotes the mass of an electron, V the volume of the system, E_F the Fermi energy and f the Fermi distribution function. The derivatives are taken in the direction of an electric field (the z direction). G is the one-particle Green function, which in the presence of a magnetic field satisfies the following equation:

$$[\![(\hbar^2/2m)\{-\mathrm{i}\nabla_r - [2\pi/(hc/e)]\boldsymbol{A}\}^2 - (E+\mathrm{i}\hbar/\tau)]\!]G(\boldsymbol{r},\boldsymbol{r}';E+\mathrm{i}\hbar/\tau) = \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{2}$$

$\boldsymbol{A}$ denotes a vector potential of the magnetic field.

In order to simplify calculations, we assume that the perpendicular cross-section of wires is smaller than the de Broglie wavelength of electrons. It reduces the problem to one dimensional on each wire. Under the present considerations, we also neglect the thermal distribution of electrons around the Fermi energy [3]. Thus the conductivity is determined for electrons exactly at the Fermi level. We wish to study the role of quantum interference in the transport process in electrical networks with different topologies and when electrons are inelastically scattered. The damping of electron waves is described by l_{in}, and this is the only parameter in this approach.

In general, the Green function G is a linear combination of independent solutions of the homogeneous differential equation (2). One can write

$$G(z,z') = [A(z')\cos(kz) + B(z')\sin(kz)]\exp(\mathrm{i}\gamma_{zz'}) \tag{3}$$

where k is a complex wavevector, i.e. $k = k' + \mathrm{i}k''$ and $E = \hbar^2 k'^2/2m$, $k'' = 1/l_{in}$. The phase shift $\gamma_{zz'} = (e/\hbar c)\int \boldsymbol{A}\cdot d\boldsymbol{l}$ results from the circulation of the vector potential $\boldsymbol{A}$ along the path of the electron between the points z and z'. The coefficients A and B are determined from the boundary conditions. The particle conservation law implies that G has to be a continuous function. Let the current source point z' lie between the node points a and b ($a \leqslant z \leqslant z' \leqslant b$). If we denote $G^<(z',z') = g_{z'}$ and $G^<(a,z') = g_a \exp(\mathrm{i}\gamma_{az'})$, then (3) can be rewritten as

$$\begin{aligned} G^<(z,z') = & \{g_a\cos(kz) + [g_{z'}\exp[(\mathrm{i}\gamma_{az'}) - g_a\cos(kz')] \\ & \times \sin(kz)/\sin(kz')\}\exp(-\mathrm{i}\gamma_{az}). \end{aligned} \tag{4}$$

The second condition is

$$\sum_i \frac{\partial G^<(z_i,z')}{\partial z_i}\bigg|_a = \delta_{a,z'} \tag{5}$$

where the summation is over all wires, which are connected to the node point a, and the derivative is taken along the wires in the direction to the node point a. The condition (5) ensures current conservation at each point of the system. The discontinuity in the

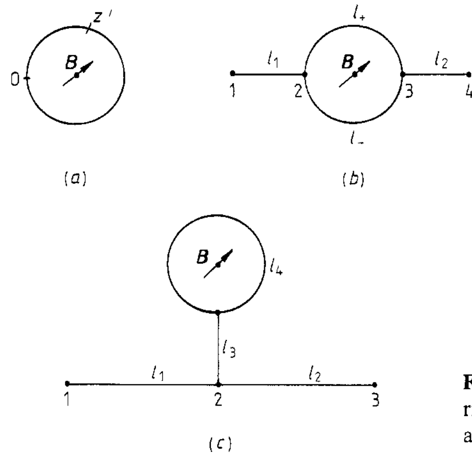

Figure 1. Systems considered: (*a*) a single ring in a magnetic field; (*b*) a ring with two attached leads; (*c*) a wire with a bubble.

derivative of the Green function at the current source point z' is included in (5). We assume free boundary conditions at the ends of wires. The above relation (5) is fulfilled in this case as well. (One could assume perfect contacts at the ends with $G(z, z') = 0$.)

Conditions (4) and (5) give us a set of equations for the functions g_a at the node points (or at the wire ends)

$$g_a \sum_b \cot(kl_{ab}) - \sum_b \frac{g_b \exp(\mathrm{i}\gamma_{ab})}{\sin(kl_{ab})} = -\frac{1}{k}\,\delta_{a,z'}. \tag{6}$$

The sum is taken over the nearest-neighbour nodes b (the current source point z' is included as well) connected to a. l_{ab} is the length of the path between the points a and b.

Solving the set (6) of linear equations, one finds g_a for all nodes and then the one-particle Green function (4) for the electrical network. It is easy to calculate the conductivity (1). One calculates the integral (1) analytically as the integrand is a linear combination of complex trigonometric functions.

3. Results

3.1. *A single ring*

An illustrative example is a ring of circumference l in a magnetic field $\boldsymbol{B}$ (figure 1(*a*)). The nodes are at the initial point O and at the current source point z'. Equations (6) for g at these points are

$$\begin{aligned} &g_0\{\cot(kz') + \cot[k(l-z')]\} - g_{z'} \exp(\mathrm{i}\varphi z')/\sin(kz') \\ &\qquad - g_{z'} \exp[-\mathrm{i}\varphi(l-z')]/\sin[k(l-z')] = 0 \\ &g_{z'}\{\cot(kz') + \cot[k(l-z')]\} - g_0 \exp(-\mathrm{i}\varphi z')/\sin(kz') \\ &\qquad - g_{z'} \exp[\mathrm{i}\varphi(l-z')]/\sin[k(l-z')] = -1/k. \end{aligned} \tag{7}$$

If the magnetic field $\boldsymbol{B}$ is perpendicular to the ring, then the magnetic flux φ enclosed in the ring is $\varphi = BS$ and the shift of the phase of the electronic wave circulating along the ring per unit length is $\varphi = 2\pi\varphi/\varphi_0 l$. ($\varphi_0 = hc/e$ and S denote the one-electron flux quantum and the area of the ring, respectively.) From (7) and (4), one obtains the Green function

$$G^{<}(z,z') = -\exp[\mathrm{i}\varphi(z-z')]\{\sin[k(l-z'+z)]+\exp(-\mathrm{i}\varphi l)\sin[k(z'-z)]\} \times \{2k[\cos(kl)-\cos(\varphi l)]\}^{-1}. \tag{8}$$

Next, we substitute G in (1) and calculate the integral. We use the symmetry relation for the Green function, which states that, if $z > z'$, then $G^{>}(z,z') = G^{<}(z',z)$. The result is

$$\sigma = (2e^2/h)l_{\mathrm{in}}\{\sinh(l/l_{\mathrm{in}})[\cosh(l/l_{\mathrm{in}})-\cos(2\pi\varphi/\varphi_0)\cos(k'l)] + \sin(2\pi\varphi/\varphi_0)\sin(k'l) \times[-\cosh(l/l_{\mathrm{in}}) + (l_{\mathrm{in}}/l)\sinh(l/l_{\mathrm{in}})]\}/|W|^2 \tag{9}$$

where

$$|W|^2 = 4|\cos(kl)-\cos(2\pi\varphi/\varphi_0)|^2 = 2\cos(2k'l)+2\cosh(2l/l_{\mathrm{in}}) - 8\cos(k'l)\cosh(l/l_{\mathrm{in}})\cos(2\pi\varphi/\varphi_0)+4\cos^2(2\pi\varphi/\varphi_0).$$

If the inelastic length is smaller than the circumference ($l_{\mathrm{in}} \ll l$), then the conductivity has the well known Boltzmann dependence $\sigma = (e^2/2h)l_{\mathrm{in}}$. In the opposite limit ($l_{\mathrm{in}} \gg l$), the smearing of electronic levels is smaller than the distance between them. There are strong resonance oscillations of the conductivity in the magnetic field. The amplitude of the oscillations is about $(e^2/2h)(l_{\mathrm{in}}^2/l)$, and the minimal value of σ is approximately $(e^2/2h)l$. Because interference processes are absent in this case (there is no scattering on impurities or contacts), the period of oscillation is $\varphi_0 = hc/e$.

3.2. *A ring with two attached leads*

Let us consider a ring with two attached leads (figure 1(*b*)). The lengths of the wire on the left-hand side and on the right-hand side of the ring are l_1 and l_2, and in the upper and the lower branch of the ring l_+ and l_-, respectively. We analyse the part of the conductance coming from the ring only, because we want to understand the role of the attached leads in the system. If the current source point z' lies on the upper branch, then equations (6) for g at the nodes on the ring are

$$\begin{aligned} &g_2[\cot(kz')+\cot(kl_-)-\tan(kl_1)] - g_{z'}\exp(\mathrm{i}\varphi z')/\sin(kz') \\ &\qquad -g_3\exp(-\mathrm{i}\varphi l_-)/\sin(kl_-) = 0 \\ &g_{z'}\{\cot(kz')+\cot[k(l_+-z')]\} - g_2\exp(-\mathrm{i}\varphi z')/\sin(kz') \\ &\qquad -g_3\exp[\mathrm{i}\varphi(l_+-z')]/\sin[k(l_+-z')] = -1/k \\ &-g_3\{\cot[k(l_+-z')]+\cot(kl_-)-\tan(kl_2)\} - g_{z'}\exp[-\mathrm{i}\varphi(l_+-z')]/\sin[k(l_+-z')] \\ &\qquad -g_2\exp(\mathrm{i}\varphi z')/\sin(kl_-) = 0. \end{aligned} \tag{10}$$

It is worth noting that, if the attached wires are very long ($l_1, l_2 \to \infty$), then they act as inelastic scatterers (in equation (10), $\tan(kl_{1,2}) \to \mathrm{i}$) (compare [3, 4]). An electron leaving the ring may spend much time in the attached wire.

Next, the Green function G and the conductance $\mathscr{G} = \sigma/l$ are calculated. Because the analytical formula is rather complicated, we present the conductance in a graphical

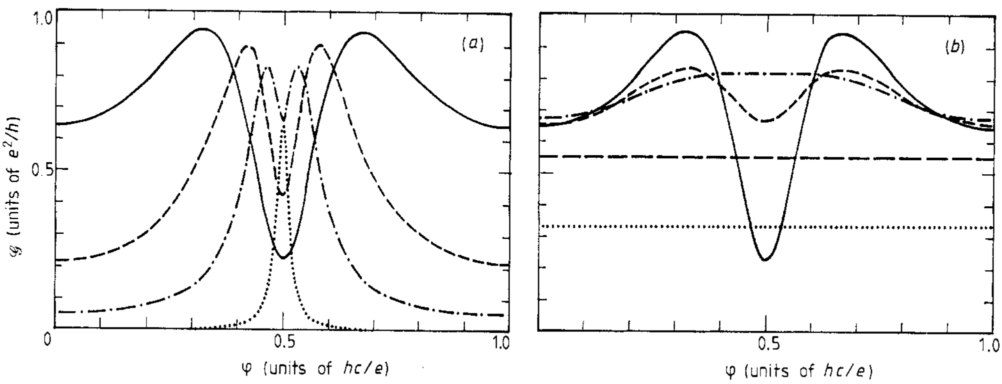

Figure 2. Dependence of the conductance $\mathcal{G}$ on the flux φ of a magnetic field for a ring with two attached leads: (*a*) $l_1 = l_2$, $l_+ = l_- = 5$, $l_{in} = 100$, $k' = \pi/2$ with different l_1 (——, $l_1 = 1000$; – – –, $l_1 = 50$; — · —, $l_1 = 20$; · · · ·, $l_1 = 0$); (*b*) $l_1 = l_2 = 1000$, $l_+ = l_- = 5$, $k' = \pi/2$ with different l_{in} (——, $l_{in} = 100$; – – –, $l_{in} = 20$; — · —, $l_{in} = 10$; — — —, $l_{in} = 2$; · · · ·, $l_{in} = 1$).

form in figure 2. Figure 2(*a*) shows the dependence of $\mathcal{G}$ on the magnetic flux φ for attached leads of various lengths. The full curve corresponds to $\mathcal{G}$ for very long leads ($l_1, l_2 \gg l_{in}$). It is very close to the transmission of an electron through the ring with two infinite leads and without inelastic scatterings obtained already previously [4]. For shorter leads ($l_1, l_2 < l_{in}$) the position of the maximum and the amplitude of $\mathcal{G}$ may be changed. It is represented in figure 2(*a*) by three other curves. We wish to point out that for the ring without leads ($l_1 = l_2 = 0$) the conductance is related to the transmission between two points (the contact points) of the ring, rather than to the conductivity along the ring (equation (9)).

Figure 2(*b*) shows the dependence of the conductance $\mathcal{G}$ on the inelastic scattering length l_{in}. With decreasing l_{in} the amplitude of oscillations of $\mathcal{G}$ decreases. The maximum of $\mathcal{G}$ may be shifted as well. One can see this for the full, broken and chain curves in figure 2(*b*), where the two maxima of $\mathcal{G}$ (for $l_{in} = 100$ and 20) are transformed into a single maximum (for $l_{in} = 10$). Thus the harmonics of oscillations change with l_{in}. The average value of $\mathcal{G}$ is almost constant up to $l_{in} \simeq l$. For l_{in} shorter than the size of the ring ($l_{in} < l_+, l_-$), the oscillations are completely damped, and the value of $\mathcal{G}$ is reduced (see two lowest curves in figure 2(*b*)).

For a different geometry of the ring (a different position of the leads attached to the ring) the shape of the conductance $\mathcal{G}$ as a function of the magnetic flux φ may be different from that presented in figure 2 as a result of different interference conditions (see also [4]). The basic period of oscillations is hc/e; however, in many cases the second harmonics (the $hc/2e$ component) dominates. In all situations, one finds familiar features for the conductance as a function of l_{in} as has been presented above.

3.3. *A wire*

Calculations for a wire are very simple. The Green function has the form

$$G^<(z, z') = \cos[k(l-z')] \cos(kz)/k \cos(kl). \quad (11)$$

Here l denotes the length of the wire. The conductance $\mathcal{G}$ is given by

$$\mathcal{G} = \sigma/l = (e^2/h)(l_{in}/l)\{\sinh(2l/l_{in}) - (l_{in}/l) \times [\coth(2l/l_{in}) - 1]\}/[\cosh(2l/l_{in}) - \cos(k'l)]. \quad (12)$$

Let us now attach two leads with a different potential level to the ends of the considered wire. For $0 \leqslant z \leqslant z' \leqslant l$ the Green function is

$$G^<(z, z') = \{\cos[k(l-z')]\cos(kz) - \mathrm{i}\delta\sin[k(l-z'+z)] + \delta^2\sin[k(l-z')]\sin(kz)\}/kW \tag{13}$$

where $W = (1+\delta)\sin(kl) + 2\mathrm{i}\delta\cos(kl)$. δ is the ratio of the Fermi wavevectors in the attached lead and in the wire, i.e. $\delta = k_0'/k' = \sqrt{(E_\mathrm{F} - V)/E_\mathrm{F}}$. We assume that the system is made up of conductors, and thus the potential step V is not too high (i.e. $E_\mathrm{F} - V > 0$). The conductance can be expressed by

$$\mathscr{G} = (e^2/h)(l_\mathrm{in}/4l)[\![[4\delta^2 + (1+\delta^2)^2]\sinh(2l/l_\mathrm{in}) + 4\delta(1+\delta^2)\cosh(2l/l_\mathrm{in}) - (l_\mathrm{in}/2l)\{(1+\delta^2)^2[1-\cosh(2l/l_\mathrm{in})] - 2\delta(1+\delta^2)\sinh(2l/l_\mathrm{in})\}]\!]/|W|^2. \tag{14}$$

The conductance is related, in this case, to transmission through a small potential barrier in the presence of inelastic scatterers.

3.4. *A wire with an attached bubble*

Now we consider an interesting case: a wire with an attached bubble (figure 1(*c*)). In classical networks the bubble does not participate in the resistance. In a quantum situation, however, electron migration through the bubble plays an important role in conductivity, only if wave coherence (which is destroyed during a scattering process) is maintained for longer than the time which a particle spends in the attached object (l_in is larger than the size of the bubble). A similar system has been investigated experimentally in [5]. Oscillations of magnetoconductance with the period hc/e were observed in [5].

In order to analyse this problem, we proceed as described above. For the points $z \leqslant z'$ on the left-hand side of the wire the Green function is

$$G^<(z, z') = \{\cos[k(l_1 - z')] + a_{234}\sin[k(l_1 - z')]\}\cos(kz)/kW \tag{15}$$

where

$$W = \sin(kl_1) - a_{234}\cos(kl_1)$$

$$a_{234} = -\tan(kl_2) + \cot(kl_3) - [\![\sin(kl_3)\{\cos(kl_3) + 2[\cos(kl_4) - \cos(2\pi\varphi/\varphi_0)]\sin(kl_3)/\sin(kl_4)\}]\!]^{-1}.$$

l_1 and l_2 are the lengths of wire on the left-hand side and the right-hand side of the bubble, l_4 is the circumference of the bubble and l_3 is the length of the lead connecting the bubble and the wire. φ denotes the magnetic flux enclosed in the bubble. One can determine the conductance as

$$\mathscr{G} = (e^2/h)[l_\mathrm{in}/2(l_1+l_2)^2][\![l_1[(1+|a_{234}|^2)\sinh(2l_1/l_\mathrm{in}) - 4\,\mathrm{Im}\,a_{234}\cosh(2l_1/l_\mathrm{in})] - l_\mathrm{in}\{(1+|a_{234}|^2) \times [\cosh(2l_1/l_\mathrm{in}) - 1] - \mathrm{Im}\,a_{234}\sinh(2l_1/l_\mathrm{in}) + \tfrac{1}{2}[1-\cosh(2l_1/l_\mathrm{in})] \times [1-\cosh(2l_2/l_\mathrm{in})]/|\cos(kl_2)|^2\} + \text{terms } (l_1 \leftrightarrow l_2)]\!]/|W|^2. \tag{16}$$

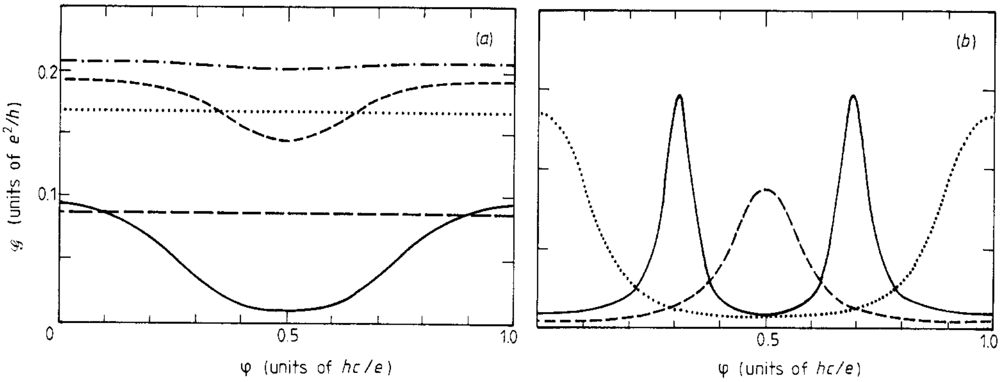

Figure 3. Dependence of the conductance $\mathscr{G}$ on the flux φ of a magnetic field for a wire with a bubble: (*a*) $l_1 = l_2 = l_3 = l_4 = 10$, $k' = \pi/2$ with different l_{in} (——, $l_{in} = 100$; - - -, $l_{in} = 20$; — · —, $l_{in} = 10$; · · · ·, $l_{in} = 5$; — — —, $l_{in} = 2$); (*b*) $l_1 = l_2 = 10$, $l_3 = 0$, $l_{in} = 100$, $k' = \pi/2$ with different l_4 (——, $l_4 = 10$; — — —, $l_4 = 16$; · · · ·, $l_4 = 30$).

Figure 3(*a*) presents the conductance as a function of φ for different l_{in}. The amplitude of oscillations of $\mathscr{G}$ is reduced with decreasing l_{in}. It is also seen that for large l_{in} the average value of $\mathscr{G}$ increases. It results from the broadening of levels and the increase in the density of states at the Fermi level, when l_{in} decreases. In the second stage (for small l_{in}) a scattering mechanism dominates and the conductivity is reduced.

The dependence of the conductance on the geometry of the system is shown in figure 3(*b*). Changing the length l_1, l_2, l_3 or l_4, one changes the distribution of energy levels and resonating conditions for transmission of electrons through the system. Therefore, the position of peaks of $\mathscr{G}$ are different in figure 3(*b*).

4. Conclusions

In this paper, we have calculated the Green functions with boundary conditions for small electrical networks and then the conductivity from the linear response theory. We investigated the influence of inelastic scattering on the quantum interference of electrons. For a very large system, when its size l is larger than the inelastic scattering length l_{in}, the conductivity is given by the Boltzmann formula. The quantum nature of the system manifests itself for small systems ($l < l_{in}$). In the quantum case, the so-called dead ends play an important role in contrast with classical networks. One can see this for a bubble attached to a wire. An experiment performed on a system with such a geometry [5] showed oscillations of the magnetoresistance with a period hc/e. We considered this theoretically. For a ring with two attached leads the shape of $\mathscr{G}$ as a function of φ depends on the geometry of the system. The basic oscillation period of $\mathscr{G}$ is hc/e, but in many cases the second harmonic (the $hc/2e$ component) dominates. It has been shown that the shape and periodicity of $\mathscr{G}$ depend on the length of inelastic scattering l_{in} as well.

We have made some simplifications, e.g. we neglected impurities. However, the method that we used is general and one can also incorporate impurities. For small systems with impurities, there is a lack of self-averaging. Cooperons as well as diffusons are not fully created and a weak localisation approach fails. We think that the present method may be used in such a case. The method can also be an alternative in investigations of tunnelling through a potential barrier, which is in contact with a thermal bath. Although we introduced *ad hoc* the damping parameter l_{in} into the Green function, it is well justified in metals (for a small potential barrier).

Acknowledgment

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